BIBLIOGRAPHY

1. Chernyi G. G., The asymptotic law of plane detonation wave propagation. Dokl. Akad. Nauk SSSR, Vol. 172, №3, 1967.
2. Levin, V. A. and Chernyi, G. G., Asymptotic laws of behavior of detonation waves. PMM Vol. 31, №3, 1967.
3. Levin, V. A., On the transition of a supercompressed plane detonation wave to the Chapman-Jouguet mode. Izv. Akad. Nauk SSSR, MZhG, №2, 1968.
4. Zaidel', R. M. and Zel'dovich, Ia. B., One-dimensional instability and attenuation of detonation. PMTF, №6, 1963.
5. Il'kaeva, L. A. and Popov, N. A., Hydrodynamic solution of one-dimensional perturbations of an unstable detonation wave. Fizika goreniia i vzryva, №3, 1965.
6. Medvedev, S. A., On the weakening of supercompressed detonation waves with finite reaction rate. Izv. Akad. Nauk SSSR, MZhG, №3, 1969.

Translated by J. J. D.

STATIONARY CONVECTION IN A VERTICAL CHANNEL WITH PERMEABLE BOUNDARIES

PMM Vol. 33, №3, 1969, pp. 476-481

G. Z. GERSHUNI, E. M. ZHUKHOVITSKII and D. L. SHVARTSBLAT
(Perm')

(Received December 11, 1968)

The problem of stationary heat convection in an infinitely long vertical flat channel with permeable boundaries is considered. The fluid is heated from below, so that in the channel there exists a constant temperature gradient. The fluid is blown into the channel through one of its vertical boundaries, and is sucked away through the other creating a transverse flow through the layer at a constant velocity. An exact solution of the problem of superposition of vertical convection on the homogeneous transverse flow is derived. Two kinds of motion are analyzed, viz. a plane, and a space motion which along the layer boundary depend periodically on the horizontal coordinate. It is shown that plane convection motions are only possible up to a certain limit of the fluid blowing-in rate.

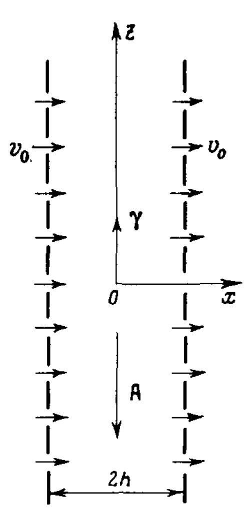

Fig. 1

1. A vertical plane layer of fluid is bounded by two parallel permeable planes $x = \pm h$ (Fig. 1). A fluid is uniformly blown into the channel through one of its boundaries at constant velocity v_0 and extracted through the other at the same uniform rate.

The heating from below generates in the fluid a vertical temperature gradient A directed downwards.

The equations of stationary convection are of the form [1]

$$(\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + g\beta T\boldsymbol{\gamma} \tag{1.1}$$

$$\operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}\nabla T = \chi\Delta T \tag{1.2}$$

Here $\mathbf{v}$ is the velocity, T the temperature, p the convection pressure, γ the unit vector directed vertically upwards, and ν, χ and β are the coefficients of kinematic viscosity, thermal diffusivity and thermal expansion, respectively.

We shall begin by considering plane motions in which the velocity is defined as the superposition of a plane-parallel convection motion on the homogeneous transversal stream

$$v_x = v_0, \qquad v_y = 0, \qquad v_z = v(x) \tag{1.3}$$

where $v_0 = \text{const}$.

We shall seek the expression of temperature and pressure distribution in the form

$$T = -Az + \theta(x), \quad p = p(z) \tag{1.4}$$

where $A = \text{const}$ is the temperature vertical gradient (*).

From Eqs. (1.1) and (1.2) we obtain

$$\nu\frac{d^2v}{dx^2} - v_0\frac{dv}{dx} + g\beta\theta = \frac{1}{\rho}\frac{dp}{dz} + g\beta Az = C' \tag{1.5}$$

$$\chi\frac{d^2\theta}{dx^2} - v_0\frac{d\theta}{dx} + Av = 0 \tag{1.6}$$

Here C' is the constant of separation of variables which defines the convective motion gradient.

We rewrite the equations of $v(x)$ and $\theta(x)$ in dimensionless form, introducing h, χ/h and Ah as the units of velocity, length and temperature respectively, and obtain

$$v'' - \frac{a}{P}v' + R\theta = C, \qquad \theta'' - a\theta' + v = 0 \tag{1.7}$$

(here the prime denotes differentiation with respect to a dimensionless coordinate).

Three dimensionless parameters, viz. the Rayleigh number R, the Péclet number a, and the Prandtl number P appear in these equations

$$R = \frac{g\beta Ah^4}{\nu\chi}, \qquad a = \frac{v_0 h}{\chi}, \qquad P = \frac{\nu}{\chi}$$

The vertical (convection) velocity v vanishes at the boundary planes, while the temperature along these varies with height according to the linear law for gradient A. For the velocity $v(x)$ and temperature $\theta(x)$ we have the homogeneous boundary conditions

$$v(\pm 1) = 0, \quad \theta(\pm 1) = 0 \tag{1.8}$$

Further to this we assume the convection stream to be closed, hence the additional condition for the convection velocity is:

$$\int_{-1}^{1} v\,dx = 0 \tag{1.9}$$

The boundary value problem (1.7)–(1.9) is an eigenvalue problem with a nontrivial solution existing for certain value of the Rayleigh number only. The characteristic values of R depend on two parameters, viz. the Péclet and the Prandtl numbers.

---

*) Solutions of the form of (1.3) and (1.4) belong to the invariant group solutions (see [2]).

2. Eliminating the unknown function $v(x)$ from system (1.7), we obtain for $\theta(x)$ one equation

$$\theta^{IV} - a\left(1 + \frac{1}{P}\right)\theta''' + \frac{a^2}{P}\theta'' - R\theta = -C \qquad (2.1)$$

The general solution of this equation is

$$\theta = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x} + C_4 e^{r_4 x} + \frac{C}{R} \qquad (2.2)$$

where r_i are the roots of the characteristic equation

$$r^4 - a\left(1 + \frac{1}{P}\right) r^3 + \frac{a^2}{P} r^2 - R = 0 \qquad (2.3)$$

The second of Eqs. (1.7) yields (2.4)

$$v = C_1 (a - r_1) r_1 e^{r_1 x} + C_2 (a - r_2) r_2 e^{r_2 x} + C_3 (a - r_3) r_3 e^{r_3 x} + C_4 (a - r_4) r_4 e^{r_4 x}$$

By satisfying the boundary conditions (1.8) and the condition of the convection stream closure (1.9) we obtain a system of five homogeneous linear algebraic equations for the coefficients C_i and constant C of the pressure gradient. Equating the determinant of this system to zero, we obtain the characteristic relationship which may be written in the form

$$(a - r_1)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4) r_1 \operatorname{cth} r_1 - (a - r_2)(r_1 - r_3)(r_1 - r_4) \times$$
$$\times (r_3 - r_4) r_2 \operatorname{cth} r_2 + (a - r_3)(r_1 - r_2)(r_1 - r_4)(r_2 - r_4) r_3 \operatorname{cth} r_3 -$$
$$- (a - r_4)(r_1 - r_2)(r_1 - r_3)(r_2 - r_3) r_4 \operatorname{cth} r_4 = 0 \qquad (2.5)$$

This relationship represents the equation used for the determination of the spectrum of eigenvalues R depending on parameters a and P. In the limit case of $a = 0$ (absence of fluid flow through the channel boundaries) the solution of this problem is known [3 and 4]. In this case the boundary value problem (1.7)–(1.9) has both even and odd solutions relative to the layer axis of symmetry, and the spectrum of eigenvalues R is independent of the Prandtl number, and consists of two subsystems each corresponding to solutions of different parity.

The eigenvalues corresponding to odd solutions are determined from the relationship

$$\sin R^{1/4} = 0, \quad R = \pi^4, \quad 16\pi^4, \ \ldots \qquad (2.6)$$

while those corresponding to even solutions are determined from equations

$$\operatorname{tg} R^{1/4} = \operatorname{th} R^{1/4}, \quad R = 237.8, \ 2497, \ \ldots \qquad (2.7)$$

For $a \neq 0$ the solution of this problem has no definite parity. In order to find the eigenvalues it is convenient in this case to rewrite Eq. (2.5) in the real form. The characteristic Eq. (2.3) has two real roots r_1 and r_2, and two complex conjugate roots $r_3 = p + iq$ and $r_4 = p - iq$. After transformation we obtain from (2.5)

$$q\left[(a - r_1)(A_2^2 + q^2) r_1 \operatorname{cth} r_1 - (a - r_2)(A_1^2 + q^2) r_2 \operatorname{cth} r_2\right] +$$
$$+ \frac{r_1 - r_2}{\operatorname{ch} 2p - \cos 2q} \{q (p \operatorname{sh} 2p + q \sin 2q) [(a - p)(A_1 + A_2) - (A_1 A_2 - q^2)] +$$
$$+ (q \operatorname{sh} 2p - p \sin 2q) [(a - p)(A_1 A_2 - q^2) + q^2 (A_1 + A_2)]\} = 0 \qquad (2.8)$$

$$A_1 = r_1 - p, \qquad A_2 = r_2 - p$$

Expression (2.8) is greatly simplified when $P = 1$. In this case the roots of Eq. (2.3) are readily found

$$r_{1,2} = {}^1/_2 (a \pm \alpha), \qquad r_{3,4} = {}^1/_2 (a \pm i\beta)$$
$$\alpha = \sqrt{4R^{1/2} + a^2}, \qquad \beta = \sqrt{4R^{1/2} - a^2}$$

The substitution of these values into (2.8) yields equation

$$\frac{\beta \operatorname{sh} \alpha}{\operatorname{ch} a - \operatorname{ch} \alpha} + \frac{\alpha \sin \beta}{\operatorname{ch} a - \cos \beta} = 0 \qquad (2.9)$$

Equation (2.8) with values of r_i obtained from (2.3) was solved numerically for various values of the Prandtl number. The four lower branches of the $R\ (a)$ spectrum for $P = 0.5$ and 1.5 are shown in Fig. 2. All branches emanate from values defined by formulas (2.6) and (2.7) for $a = 0$ (absence of fluid blowing-in). The most outstanding characteristic of the obtained spectrum of eigenvalues is the pairwise closing of adjacent branches with increasing Péclet number. Since a stationary convection motion in the configuration here considered (1.3) and (1.4) exists only for parameter values corresponding to branches of the $R\ (a)$ spectrum, the occurence of branch "closing" clearly means that at sufficiently high rates of blowing-in the considered convection pattern is not possible.

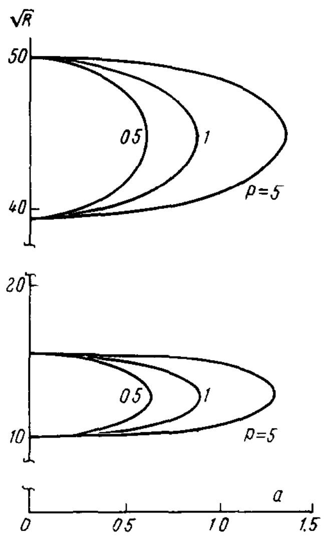

Fig. 2

Limit values of the Péclet number a_* are comparatively small.

Thus, for example, for $P = 1$ we have $a_* = 0.88$. Therefore, in the region where $a > a_*$ a plane-parallel convection on the background of a homogeneous transversal stream is not possible.

This, of course, does not mean that in the region $a > a_*$ a nonconvective pattern of the transverse flow only is possible; the existence of stationary convection of a different pattern in this region cannot be excluded.

3. We shall now consider spatial vertical flows periodic along the horizontal coordinate y (the y-axis is normal to the drawing plane of Fig. 1).

We shall look for the solution of the stationary convection equation in the form

$$v_x = v_0, \quad v_y = 0, \qquad v_z = v\,(x) \cos ky$$

$$T = -Az + \theta\,(x) \cos ky, \qquad p = p\,(z) \qquad (3.1)$$

Here k is the wave number defining the periodicity along y.

Retaining the same units as in Sect. 1 we obtain the following dimensionless equations:

$$(v'' - k^2 v) - \frac{a}{P} v' + R\theta = 0 \qquad (3.2)$$

$$(\theta'' - k^2\theta) - a\theta' + v = 0 \qquad (3.3)$$

with boundary conditions

$$v\,(\pm 1) = 0, \quad \theta\,(\pm 1) = 0 \qquad (3.4)$$

(k is the dimensionless wave number).

It should be noted that a motion of the form defined by (3.1) automatically ensures the fulfilment of the vertical stream closure condition due to the periodic dependance of the convection velocity on y; the additional condition (1.9) is now redundant. The constant of separation of variables in Eq. (3.2) bound with the pressure gradient is for the same reason equal to zero.

The homogeneous boundary value problem (3.2)–(3.4) leads to the characteristic equation defining the spectrum of eigenvalues $R\ (a,\ P,\ k)$

$$(u_1u_2 + u_3u_4)\ \mathrm{sh}\ (r_1 - r_2)\ \mathrm{sh}\ (r_3 - r_4) -$$
$$- (u_1u_3 + u_2u_4)\ \mathrm{sh}\ (r_1 - r_3)\ \mathrm{sh}\ (r_2 - r_4) +$$
$$+ (u_1u_4 + u_2u_3)\ \mathrm{sh}\ (r_1 - r_4)\ \mathrm{sh}\ (r_2 - r_3) = 0 \qquad (3.5)$$

Here $u_i = r_i\ (a - r_i) + k^2$, and r_i are the roots of equation

$$r^4 - a\left(1 + \frac{1}{P}\right) r^3 + \left(\frac{a^2}{P} - 2k^2\right) r^2 + ak^2\left(1 + \frac{1}{P}\right) r - (R - k^4) = 0 \quad (3.6)$$

In the particular case of $P = 1$ the spectrum is simple. From (3.6) we obtain

$$r_{1,2} = {}^1/_2\,[a \pm \sqrt{4\,(R^{1/2} + k^2) + a^2}], \qquad r_{3,4} = {}^1/_2\,[a \pm i\,\sqrt{4\,(R^{1/2} - k^2) - a^2}] \quad (3.7)$$

Substituting these values into (3.5) we obtain branches of the spectrum of R

$$R = {}^1/_{16}\,(n^2\pi^2 + 4k^2 + a^2)^2, \qquad n = 1, 2, 3, \ldots \quad (3.8)$$

At the limit $a = 0$ (absence of percolation) we obtain the spectrum derived in [5].

It will be seen from formula (3.8) that in the case of spatial flows the characteristic values of R increase monotonically with increasing Péclet number; "closures" of the $R\ (a)$ levels are absent. Thus, the spectra of eigenvalues R differ substantially for plane and spatial motions.

4. The problem analyzed in the foregoing is closely related to the problem of convection stability in a fluid heated from below. If a homogeneous transverse flow is taken as the unperturbed state, and its stability with respect to convection induced by heating from below is analyzed by the method of small perturbations, then for the amplitudes of small plane-parallel perturbations we obtain the boundary value problem (1.7)–(1.9), and for spatial perturbation we have problem (3.2)–(3.4). Thus the problem of stationary convection coincides with that of critical perturbation which, as is known, is typical for vertical channels. The eigenvalues R determined above have, with respect to stability of stationary transverse flow, also the significance of critical values. A transverse flow passing through critical values of R_i (with increasing Rayleigh number) becomes unstable with respect to the next following vertical convection perturbation mode. Hence, curves of $R\ (a)$ represent neutral lines of stationary perturbations.

The closure of instability levels in the case of plane perturbations (Fig. 2) means that an increase of the transverse flow rate results in the establishment of complete stability. A similar closure of lines of the instability spectrum was described in [6] which dealt with the stability of stationary convection motion in an inclined layer. It was shown there that the closure of convection instability modes is under certain conditions accompanied by the appearance of oscillatory instability. It would be reasonable to think that also in the problem considered above the stationary oscillation kind of motions is possible. A complete clarification of this question necessitates the analysis of the nonstationary plane-parallel motions. Such an analysis is at present in progress.

In the case of spatial perturbations the critical Rayleigh numbers increase monotonically with increasing parameter a. The stability with respect to spatial perturbations increases, but an absolute stability is not reached.

In concluding we note that the effect of transverse seepage of fluid on convection in a horizontal layer heated from below had been previously investigated [7]. The transverse motion in a horizontal layer also leads to increased stability. The critical Rayleigh numbers increase monotonically with increasing Peclet number; the closing of levels is in this case absent. There is thus a similarity with spatial perturbations in a vertical layer. However, when comparing the results of [7] with those derived here, it should be stressed that there is no complete analogy between the two problems. In the case of the horizontal layer the transverse motion is directed across the unperturbed isotherms resulting in the decrease of the unstably stratified layer thickness with increasing velocity of the transverse motion. The transverse motion in a vertical layer occurs, on the other hand, along the isotherms without distorting the temperature distribution equilibrium.

BIBLIOGRAPHY

1. Landau, L. D. and Lifshits, E. M., Mechanics of Continuous Media. M., Gostekhizdat, 1954.
2. Katkov, V. L., Exact solutions of certain convection problems. PMM Vol. 32, №3, 1968.
3. Ostroumov, G. A., Free Convection in Conditions of the Internal Problem. M.-L., Gostekhizdat, 1952.
4. Chia-Shun Jih, Thermal instability of viscous fluids. Quart. appl. math., Vol. 17, №1, 1959.
5. Wooding, R. A., Instability of a viscous liquid of variable density in a vertical Hele-Shaw cell. J. Fluid mech., Vol. 7, №4, 1960.
6. Birikh, R. V., Gershuni, G. Z., Zhukhovitskii, E. M. and Rudakov, R. N., Hydrodynamic and thermal instability of a steady convective flow. PMM Vol. 32, №2, 1968.
7. Shvartsblat, D. L., The spectrum of perturbations and convective instability of a plane horizontal fluid layer with permeable boundaries. PMM Vol. 32, №2, 1968.

Translated by J. J. D.

A HYDRODYNAMIC MODEL OF DISPERSE SYSTEMS

PMM Vol. 33, №3, 1969, pp. 482-494

Iu. A. BUEVICH
(Moscow)

(Received December 19, 1968)

The dynamic equations of motion of the phases of a monodisperse system are formulated in the approximation of interpenetrating interacting continua. The energy transfer equations of the pulsations of the phases in various directions are derived. These serve to close the above system of dynamic equations.

We investigated the non-Newtonian hydromechanics of disperse systems in [1] and extended it to gas suspensions in [2, 3]. The approach used in [1–3] is to some extent phenomenological, in that the random pulsations of the phases of a disperse system are dealt with on the basis of the equations of motion of the phases postulated *a priori* as for continua, whereas strictly speaking such equations can only be posited without contradiction after such analysis. We now propose to eliminate the contradiction by